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Spectra of non-commutative dynamical systems and graphs related to fractal groups

Laurent BARTHOLDI, Rostislav I. GRIGORCHUK

Section de Mathématiques, Université de Genève, CP 240, 1211 Genève 24, Suisse
E-mail address: Laurent.Bartholdi@math.unige.ch

Steklov Institute of Mathematics, Gubkina 8, Moscow 117966, Russia
E-mail address: grigorch@mi.ras.ru

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Abstract. We study spectra of noncommutative dynamical systems, representations of fractal groups, and regular graphs. We explicitly compute these spectra for five examples of groups acting on rooted trees, and in three cases obtain totally disconnected sets.

Spectres de systèmes dynamiques non-commutatifs et de graphes associés à des groupes fractals

Résumé. Nous étudions les spectres de systèmes dynamiques non commutatifs, de représentations de groupes fractals, et de graphes réguliers. Nous calculons ces spectres pour cinq exemples de groupes agissant sur des arbres enracinés, et dans trois cas obtenons des ensembles complètement déconnectés.

Version française abrégée

Le but de cette note est d'étudier le spectre de systèmes dynamiques non commutatifs, i.e. de systèmes engendrés par plusieurs transformations qui ne commutent pas nécessairement. Elle résume les résultats de [1].

Soit G un groupe engendré par un ensemble symétrique fini S. Un *système dynamique*, noté (S, X, μ), est une action de G sur un espace mesuré X préservant la classe $[\mu]$ de la mesure μ de X. Il induit naturellement une représentation unitaire π de G dans $L^2(X, \mu)$ donnée par $(\pi(g)f)(x) = \sqrt{\mathfrak{g}(x)}f(g^{-1}x)$, où $\mathfrak{g}(x)$ est la dérivée de Radon-Nikodým $dg\mu(x)/d\mu(x)$. Le *spectre* de (S, X, μ) est le spectre de l'opérateur de type Hecke

$$H_\pi = \frac{1}{|S|}\sum_{s\in S}\pi(s) \in \mathcal{B}(L^2(X, \mu)).$$

Plus généralement, le spectre d'une représentation unitaire $\pi : G \to \mathcal{U}(\mathcal{H})$ d'un groupe avec système générateur fixé est le spectre de l'opérateur H_π comme ci-dessus.

On considère, pour cinq exemples de groupes donnés par leur action sur un arbre régulier enraciné $\mathcal{T}$, leur représentation π dans $L^2(\partial\mathcal{T}, \mu)$ où μ est la mesure de Bernoulli uniforme. Cette représentation s'approxime par les représentations π_n sur les sommets à distance n de la racine de $\mathcal{T}$.

Nous montrons que π se décompose en une somme de représentations $\pi_n \ominus \pi_{n-1}$ de dimensions finies, et $\operatorname{spec}(\pi) = \overline{\bigcup_{n\geq 0}\operatorname{spec}(\pi_n)}$.

H_π a un spectre purement ponctuel et son rayon spectral est une valeur propre. On peut décrire explicitement ces spectres comme suit. Pour $\lambda \in \mathbb{R}$, soit $J(\lambda)$ l'ensemble de Julia du polynôme quadratique $z \mapsto z^2 - \lambda : J(\lambda) = \overline{\left\{\sqrt{\lambda \pm \sqrt{\lambda \pm \sqrt{\lambda \pm \sqrt{\dots}}}}\right\}}$.

Note présentée par Jean-Pierre Serre

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Groupe	Spectre de H_π	Description
$\mathfrak{G}$	$[-\frac{1}{2},0]\cup[\frac{1}{2},1]$	deux intervalles
$\tilde{\mathfrak{G}}$	$[0,1]$	intervalle positif
Γ	$\{1,\frac{1}{4}\}\cup\frac{1}{4}(1\pm J(6))\}$	union d'un ensemble de Cantor de mesure nulle et de points isolés
$\overline{\Gamma}$	$\{1,-\frac{1}{2},\frac{1}{4}\}\cup\frac{1}{4}\Big(1\pm\sqrt{\frac{9}{2}\pm 2J(\frac{45}{16})}\Big)$	ensemble de Cantor de mesure nulle
$\overline{\overline{\Gamma}}$		identique à $\overline{\Gamma}$

Ces calculs impliquent l'existence de graphes à croissance polynomiale, qui sont les graphes de Schreier de groupes de croissance intermédiaire, et dont le spectre de l'opérateur de Markov est un quelconque des ensembles ci-dessus. Un résultat analogue est vrai pour les systèmes dynamiques.

1. Introduction

The purpose of this note is the study of spectra of noncommutative dynamical systems (that is, systems generated by several transformations that do not necessarily commute). We produce several examples of computations of such spectra with an interesting topological structure. More details appear in [1].

In the classical case defined by a single aperiodic measure-preserving transformation $T: X \to X$, the spectrum of the corresponding operator $\frac{1}{2}(U+U^{-1})$, with $U \in \mathcal{U}(L^2(X))$, is $[-1,1]$ by Rohlin's Lemma, but in the noncommutative case the spectrum may have gaps, and even be a Cantor set.

We also study spectra of infinite regular graphs and produce the first example of a regular graph whose spectrum is a Cantor set.

Our dynamical systems (with associated Hecke operator) arise from actions of fractal groups on the boundary of the regular rooted tree on which they act. The graphs (with associated Markov operator) whose spectra we consider are the Schreier graphs of these groups over "parabolic subgroups". In special cases, these graphs are "substitutional graphs" and have polynomial growth.

If the underlying group is amenable, then the spectrum of the the dynamical system coincides with the spectrum of the corresponding graph.

The computation of the above spectra is based on operator recursions that hold for fractal groups and involves a 1-dimensional and 2-dimensional classical dynamical system as an intermediate step. This leads to the appearance of Julia sets of quadratic maps in the description of the above spectra.

Both authors wish to thank heartily Pierre de la Harpe and Alain Valette for their numerous comments and contributions to this note.

2. Spectra of Dynamical Systems and Representations

Let G be a group finitely generated by a symmetric set S. A *non-commutative dynamical system*, denoted (S,X,μ), is an action of a group G (generated by S) on a space X and preserving the measure class $[\mu]$ of a measure μ on X.

Such a dynamical system gives rise to a natural unitary representation π of G in $L^2(X,\mu)$ given by

$$(\pi(g)f)(x) = \sqrt{\mathfrak{g}(x)}f(g^{-1}x),$$

where $\mathfrak{g}(x) = dg\mu(x)/d\mu(x)$ is the Radon-Nikodým derivative. The *spectrum* of the dynamical system (S,X,μ) is the spectrum of the Hecke type operator

$$H_\pi = \frac{1}{|S|}\sum_{s\in S}\pi(s) \in \mathcal{B}(L^2(X,\mu))$$

(this terminology comes from an analogy with Hecke operators in number theory [13].) More generally, the spectrum of a unitary representation $\pi: G \to \mathcal{U}(\mathcal{H})$ of a group with a given finite set of generators is the spectrum of the operator H_π as above — see [10].

Definition. *A* graph *is a pair $\mathcal{G}=(V,E)$ of sets (the* vertices *and* edges*), a map $\alpha: E\to V$ (the* start *of an edge) and an involution $\bar{\cdot}: E \to E$ (the* inversion*). One defines then the* end $\omega(e)$ *of an edge by $\omega(e)=\alpha(\overline{e})$.*

The degree *of a vertex is $\deg(v)=|\{e\in E|\,\alpha(e)=v\}|$. The graph is* locally finite *if $\deg(v)<\infty$ for all $v\in V$, and is* regular *if $\deg(v)$ is constant over V. The graph is a* tree *if in every circuit $(e_1,e_2,\dots,e_n)$ of edges with $\omega(e_i)=\alpha(e_{i+1})$ (indices modulo n) there is a* reduction*, i.e. an i with $\overline{e_i}=e_{i+1}$. A* graph morphism *is a pair of maps between the vertex and edge sets that commute with α and $\bar{\cdot}$.*

Let $\Sigma = \{1, \dots, d\}$ be a finite set of cardinality d. The d-regular rooted tree $\mathcal{T}_d$ *is the graph with vertex set Σ^*, edge set $\Sigma^* \times \Sigma \times \{\pm\}$, and maps $\alpha(\sigma, s, +) = \sigma$, $\alpha(\sigma, s, -) = \sigma s$ and $\overline{(\sigma, s, +)} = (\sigma, s, -)$. Its* boundary *is $\partial\mathcal{T} = \Sigma^{\mathbb{N}}$, the set of infinite sequences over Σ.*

Suppose now that G acts by automorphisms on a rooted tree $\mathcal{T}$. This action extends to a continuous action on the boundary of the tree, which is a compact, totally disconnected space. The *uniform measure* on $\Sigma^{\mathbb{N}}$ is the measure μ defined on the cylinders $\sigma\Sigma^{\mathbb{N}}$ (for $\sigma \in \Sigma^*$) by $\mu(\sigma\Sigma^*) = d^{-|\sigma|}$. It is G-invariant, and is the unique invariant measure if G acts transitively on each level Σ^n of the tree, or equivalently if $(S, \Sigma^{\mathbb{N}}, \mu)$ is ergodic. Note that all other nondegenerate Bernoulli measures are quasi-invariant.

For each $n \in \mathbb{N}$, let π_n be the unitary representation (of finite dimension d^n) of G in $\ell^2(\Sigma^n)$ induced by the action of G on the n-th level, and let π be the unitary representation of G in $L^2(\Sigma^{\mathbb{N}}, \mu)$. Then π_n is a subrepresentation of π_{n+1} and of π, and the spectra of π_n converge to that of π:

$$\operatorname{spec}(\pi) = \overline{\bigcup_{n\in\mathbb{N}} \operatorname{spec}(\pi_n)}.$$

3. Spectra of Graphs

Let $\mathcal{G}$ be a locally finite graph. The *Markov operator* of $\mathcal{G}$ is the operator M on $\ell^2(V)$ given by

$$(Mf)(v) = \frac{1}{\deg v} \sum_{e\in E:\, \alpha(e)=v} f(\omega(e)).$$

The operator M is the transition operator for the simple random walk on $\mathcal{G}$.

The spectral properties of M are of great importance; for instance, a theorem of Kesten [11] (extended by Dodziuk and others) claims that a graph $\mathcal{G}$ of bounded degree is amenable if and only if $1 \in \operatorname{spec}(M)$. (See more on amenability in Section 5.)

Definition. *Let G be a group finitely generated by a symmetric set S, and let H be any subgroup. The* Schreier graph *of G with respect to H is the graph $\mathcal{S}(G, H, S)$ with vertex set G/H and edge set $S \times G/H$, and maps $\alpha(s, gH) = gH$ and $\overline{(s, gH)} = (s^{-1}, sgH)$. It has a natural base-point H.*

In Subsection 4.3 the graphs will be labelled. This is simply done by assigning to each edge (s, gH) the labeling s.

If $H = 1$, we obtain the usual Cayley graph of (G, S). Note that $\mathcal{S}(G, H, S)$ is an $|S|$-regular graph, but its automorphism group does not necessarily act transitively on its vertices. Indeed, basically any regular graph is a Schreier graph [12, Theorem 5.4].

G acts by left-multiplication on G/H, the vertex set of $\mathcal{S}(G, H, S)$. The corresponding unitary representation in $\ell^2(G/H)$ is the *quasi-regular* representation $\rho_{G/H}$.

Suppose now that G acts on a rooted tree $\mathcal{T}_d$. Fix the ray $e = dd\cdots \in \Sigma^{\mathbb{N}}$. Let P_n be the stabilizer of vertex at level n of this ray, and let $P = \bigcap_{n\in\mathbb{N}} P_n$ be the stabilizer of the infinite ray e (it is called a *parabolic subgroup*). We write M_n the Markov operator of $\mathcal{S}(G, P_n, S)$ and M the Markov operator of $\mathcal{S}(G, P, S)$; they are the Hecke type operators associated with the action of G on Σ^n and $\Sigma^{\mathbb{N}}$.

4. Fractal Groups and Substitutional Graphs

Let G be a group acting on a tree $\mathcal{T}$, and let $H = \bigcap_{s\in\Sigma} \mathsf{Stab}_G(s)$ be the *first level stabilizer*. Restricting the action of H to each subtree spanned by $s\Sigma^*$ gives an embedding

$$\psi : H \to \mathsf{Aut}(\mathcal{T})^{\Sigma}.$$

Definition. *The group G is* fractal *if ψ is a subdirect embedding of H in G^{Σ}, i.e. if $\psi(H)$ lies in G^{Σ} and its projection on each factor is onto.*

Let now G be a group finitely generated by a symmetric set S, and let (X, x_0) be a pointed space on which G acts. The *growth* of X is the function $\gamma : \mathbb{N} \to \mathbb{N}$ given by

$$\gamma(n) = |\{x \in X : x = s_1 \dots s_n x_0 \text{ for some } s_i \in S\}|.$$

X has *polynomial growth* if $\gamma(n) \leq n^D$ for some large enough D, has *exponential growth* if $\gamma(n) \geq \lambda^n$ for some small enough $\lambda > 1$, and has *intermediate growth* in the remaining cases. The growth of G is its growth under its action on itself by left multiplication.

We describe now briefly five archetypical examples of groups.

4.1. **The Groups $\mathfrak{G}$ and $\tilde{\mathfrak{G}}$.** The first group was introduced by the second author in 1980 [5]; both groups act on $\mathcal{T}_2$. Let a be the automorphism permuting the top two branches of $\mathcal{T}_2$, and define recursively b, c, d by $\phi(b) = (a, c)$, $\phi(c) = (a, d)$ and $\phi(d) = (1, b)$. Let $\mathfrak{G}$ be the group generated by $\{a, b, c, d\}$.

Define also $\tilde{b}, \tilde{c}, \tilde{d}$ by $\phi(\tilde{b}) = (a, \tilde{c})$, $\phi(\tilde{c}) = (1, \tilde{d})$ and $\phi(\tilde{d}) = (1, \tilde{b})$, and let $\tilde{\mathfrak{G}}$ be the group generated by $\{a, \tilde{b}, \tilde{c}, \tilde{d}\}$. Clearly $\tilde{\mathfrak{G}}$ contains $\mathfrak{G}$ as the subgroup $\langle a, \tilde{b}\tilde{c}, \tilde{c}\tilde{d}, \tilde{d}\tilde{b}\rangle$.

4.2. *GGS* **Groups.** Let d be a prime number. Denote by a the automorphism of $\mathcal{T}_d$ permuting cyclically the top d branches. Fix a sequence $\epsilon = (\epsilon_1, \dots, \epsilon_{d-1}) \in (\mathbb{Z}/d)^{d-1}$. Define recursively the automorphism t_ϵ of $\mathcal{T}_d$, written $t_\epsilon = (a^{\epsilon_1}, \dots, a^{\epsilon_{d-1}}, t_\epsilon)$, by

$$t_\epsilon(\sigma_1\sigma_2\dots\sigma_n) = \begin{cases} \sigma_1 a^{\epsilon_{\sigma_1}}(\sigma_2\dots\sigma_n) & \text{if } 1 \le \sigma_1 \le d-1, \\ \sigma_1 t_\epsilon(\sigma_2\dots\sigma_n) & \text{if } \sigma_1 = d. \end{cases}$$

Then G_ϵ is the subgroup of $\mathsf{Aut}(\mathcal{T}_d)$ generated by $\{a, t_\epsilon\}$. It is called a *GGS group* [3].

The following results belong to folklore (see [6]): G_ϵ is an infinite group if an only if $\epsilon \neq (0, \dots, 0)$. It is a torsion group if and only if $\sum \epsilon_i = 0$. The only three infinite *GGS* groups for $d = 3$ are as follows:

Let r be the automorphism of $\mathcal{T}_3$ defined recursively by $\phi(r) = (a, 1, r)$, and let Γ be the subgroup of $\mathsf{Aut}(\mathcal{T}_3)$ generated by $\{a, r\}$. It was first considered by Narain Gupta and Jacek Fabrykowski [4].

Define s by $\phi(s) = (a, a, s)$, and let $\overline{\Gamma}$ be the subgroup of $\mathsf{Aut}(\mathcal{T}_3)$ generated by $\{a, s\}$.

Define t by $\phi(t) = (a, a^{-1}, t)$, and let $\overline{\overline{\Gamma}}$ be the subgroup of $\mathsf{Aut}(\mathcal{T}_3)$ generated by $\{a, t\}$; it was first studied in the 80's by Narain Gupta and Said Sidki [8, 9].

Theorem 1. *The groups $\mathfrak{G}, \tilde{\mathfrak{G}}$ and all infinite GGS groups with $d \ge 3$ have intermediate growth. The Schreier graph $\mathcal{S}(G, P, S)$ corresponding to $G = \mathfrak{G}, \tilde{\mathfrak{G}}$ or any GGS group has polynomial growth.*

In particular, $\mathcal{S}(\mathfrak{G}, P, S)$ and $\mathcal{S}(\tilde{\mathfrak{G}}, P, S)$ have linear growth, while $\mathcal{S}(\Gamma, P, S)$, $\mathcal{S}(\overline{\Gamma}, P, S)$ and $\mathcal{S}(\overline{\overline{\Gamma}}, P, S)$ have polynomial growth of degree $\log_2(3)$.

4.3. **Substitutional Graphs.** We give a self-contained description of the Schreier graphs $\mathcal{S}(G, P, S)$ for the examples above, in the form of *substitutional rules*. In this subsection, all graphs shall have a base point, and shall be edge-labelled; graph embeddings must preserve the labelings.

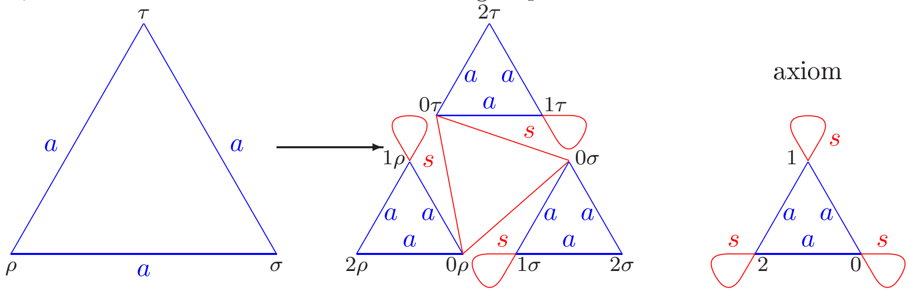
Definition. *A* substitutional rule *is a tuple $(U, R_1, \dots, R_n)$, where U is a finite d-regular edge-labelled graph, called the* axiom*, and each R_i is a rule of the form $X_i \to Y_i$, where X_i and Y_i are finite edge-labelled graphs. The graphs X_i are required to have no common label. Furthermore, there is an inclusion, written ι_i, of the vertices of X_i in the vertices of Y_i; the degree of $\iota_i(x)$ is the same as the degree of x for all $x \in V(X_i)$, and all vertices of Y_i not in the image of ι_i have degree d.*

Given a substitutional rule, one sets $\mathcal{G}_0 = U$ and constructs iteratively $\mathcal{G}_{n+1}$ from $\mathcal{G}_n$ by listing all embeddings of all X_i in $\mathcal{G}_n$ (noting that they are disjoint), and replacing them by the corresponding Y_i. If the base point $*$ of $\mathcal{G}_n$ is in a graph X_i, the base point of $\mathcal{G}_{n+1}$ will be $\iota_i(*)$.

Note that this expansion operation preserves the degree, so $\mathcal{G}_n$ is a d-regular finite graph for all n. We are interested in fixed points of this iterative process, or equivalently in a converging sequence of balls of increasing radius in the $\mathcal{G}_n$, and call a limit graph (which exists by [7]) a *substitutional graph*.

Theorem 2. *For the five examples G described above, the Schreier graphs $\mathcal{S}(G, P, S)$ are substitutional graphs.*

As an illustration, here is the substitutional rule for the group Γ:



5. Amenability and Spectra

Let G be a group acting on a set X. This action is *amenable* in the sense of von Neumann [14] if there exists a finitely additive measure μ on X, invariant under the action of G, with $\mu(X)=1$.

A group G is *amenable* if its action on itself by left-multiplication is amenable.

We now state the main connection between the spectra of our representations and dynamical systems. Recall π is the representation of G on $L^2(\Sigma^{\mathbb{N}})$ and π_n is the representation of G on $L^2(\Sigma^n)$. Since π_n contains π_{n-1} we write $\pi_n=\pi_{n-1}\oplus\pi_n^\perp$.

Theorem 3. *Let G be a group acting on a regular rooted tree, with π, π_n and $\pi_n^\perp$ be as above.*

1. *π is a reducible representation of infinite dimension but splits as $\pi_0\oplus\bigoplus_{n\geq 1}\pi_n^\perp$, so all of its irreducible components are finite-dimensional. Moreover $\operatorname{spec}(\pi)=\overline{\bigcup_{n\geq 0}\operatorname{spec}(\pi_n)}$.*

   However, if G is weak branch (see [2]), then $\rho_{G/P}$ is irreducible.
2. *The representations π_n and ρ_{G/P_n} are equivalent, so their spectra coincide. The spectrum of $\rho_{G/P}$ is contained in $\overline{\cup_{n\geq 0}\operatorname{spec}(\pi_n)}$, and therefore is contained in the spectrum of π.*

   If moreover either P or G/P are amenable, these spectra coincide, and if P is amenable, they are contained in the spectrum of ρ_G.
3. *H_π has a pure-point spectrum, and its spectral radius $r(H_\pi)=s\in\mathbb{R}$ is an eigenvalue, while the spectral radius $r(H_{\rho_{G/P}})$ is not an eigenvalue of $H_{\rho_{G/P}}$. Therefore $H_{\rho_{G/P}}$ and H_π are different operators having the same spectrum.*

6. Results on Spectra

Since all the groups considered have intermediate growth, they are amenable and $\operatorname{spec}\rho_{G/P}=\operatorname{spec}\pi$. We now describe explicitly these spectra. For $\lambda\in\mathbb{R}$ let $J(\lambda)$ be the Julia set of the quadratic map $z\mapsto z^2-\lambda$:

$$J(\lambda)=\overline{\left\{\sqrt{\lambda\pm\sqrt{\lambda\pm\sqrt{\lambda\pm\sqrt{\dots}}}}\right\}}.$$

Group	Spectrum of M	Description
$\mathfrak{G}$	$[-\frac{1}{2},0]\cup[\frac{1}{2},1]$	two intervals
$\tilde{\mathfrak{G}}$	$[0,1]$	nonnegative interval
Γ	$\{1,\frac{1}{4}\}\cup\frac{1}{4}(1\pm J(6))\}$	union of Cantor set of null Lebesgue measure and set of isolated points
$\overline{\Gamma}$	$\{1,-\frac{1}{2},\frac{1}{4}\}\cup\frac{1}{4}\left(1\pm\sqrt{\frac{9}{2}\pm 2J(\frac{45}{16})}\right)$	Cantor set of null Lebesgue measure
$\overline{\overline{\Gamma}}$		same as for $\overline{\Gamma}$

These computations imply the following results:

Theorem 4. 1. *There are connected 4-regular graphs of polynomial growth, which is are the Schreier graphs of groups of intermediate growth, and whose Markov operator's spectrum is any of the above sets.*

2. *There are noncommutative dynamical systems generated by 3 (in the case of $\mathfrak{G}$), 4 (in the case of $\tilde{\mathfrak{G}}$) or 2 transformations, whose spectrum is any of the above sets.*

These spectra are all computed using the same technique: consider the representation π_n of dimension d^n, given by $d^n\times d^n$-permutation matrices $\pi_n(s)$, for all $s\in S$. These matrices satisfy block identities, for instance for the group $\mathfrak{G}$

$$\pi_n(a)=\begin{bmatrix}0 & 1_{d^{n-1}}\\ 1_{d^{n-1}} & 0\end{bmatrix},\qquad \pi_n(b)=\begin{bmatrix}\pi_{n-1}(a) & 0\\ 0 & \pi_{n-1}(c)\end{bmatrix},$$
$$\pi_n(c)=\begin{bmatrix}\pi_{n-1}(a) & 0\\ 0 & \pi_{n-1}(d)\end{bmatrix},\qquad \pi_n(d)=\begin{bmatrix}1_{d^{n-1}} & 0\\ 0 & \pi_{n-1}(b)\end{bmatrix}.$$

Note that for our five examples $\pi_n(s)$ is expressed by blocks of the form 0, 1 and $\pi_{n-1}(s')$.

Define now the polynomial $Q:\mathbb{C}^{S+1}\to\mathbb{C}$ by

$$Q_n(\{X_s\},\lambda)=\det\left(\sum_{s\in S}X_s\pi_n(s)-\lambda\right).$$

The spectrum of π_n is $\{\lambda \in \mathbb{C} | Q_n(\frac{1}{|S|}, \dots, \frac{1}{|S|}, \lambda) = 0\}$. Using the above block identities, it is possible to express Q_n as rational expression over Q_{n-1}, and therefore to compute inductively Q_n and the spectrum of π_n. For the group $\mathfrak{G}$, for instance, we have for $n \geq 2$, setting $\alpha = X_a$ and $\beta = X_b = X_c = X_d$,

$$Q_n(\alpha, \beta, \lambda) = (3\beta^2 + 2\lambda\beta - \lambda^2)^{2^{n-2}} Q_{n-1}\left(\frac{2\alpha^2}{2\beta^2\lambda - \lambda^2}, \lambda - \beta + \frac{(\lambda - \beta)\alpha^2}{3\beta^2 + 2\beta\lambda - \lambda^2}\right).$$

In all our cases the polynomials Q_n can be explicitly computed; again for the group $\mathfrak{G}$, we have the

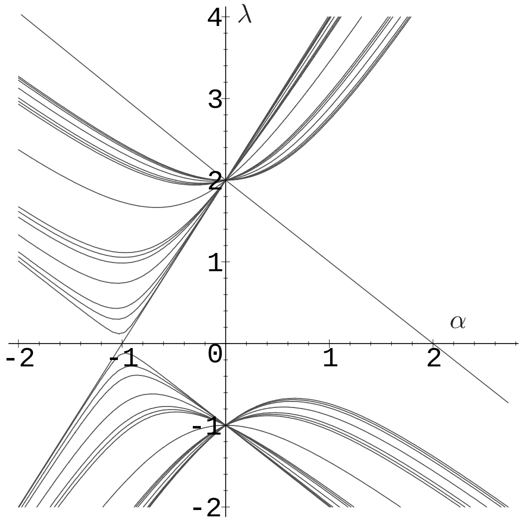
Lemma 1. *Define*

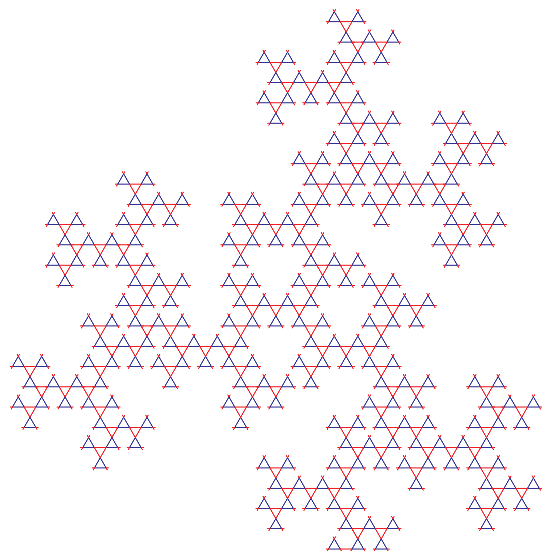
$$\Phi_0 = \alpha + 3\beta - \lambda, \qquad \Phi_1 = -\alpha + 3\beta - \lambda,$$

$$\Phi_2 = -\alpha^2 - 3\beta^2 - 2\beta\lambda + \lambda^2, \qquad \Phi_n = \Phi_{n-1}^2 - 2(2\alpha)^{2^{n-2}} \text{ for } n \geq 3.$$

Then for all $n \in \mathbb{N}$ we have $Q_n(\alpha, \beta, \lambda) = \Phi_0 \Phi_1 \cdots \Phi_n$.

We show below the set of vanishing points of $Q_6(X_a = X_{a^{-1}} = \alpha, X_r = X_{r^{-1}} = 1, \lambda)$ for the group Γ, and the Schreier graph $\mathcal{S}(\Gamma, P_6, \{a^{\pm 1}, r^{\pm 1})$.





On montre ci-dessus l'ensemble des points d'annulation de $Q_6(X_a = X_{a^{-1}} = \alpha, X_r = X_{r^{-1}} = 1, \lambda)$ pour le groupe Γ, et le graphe de Schreier $\mathcal{S}(\Gamma, P_6, \{a^{\pm 1}, r^{\pm 1})$.

References

1. Laurent Bartholdi and Rostislav I. Grigorchuk, *On the spectrum of Hecke type operators related to some fractal groups*, to appear in Proc. Steklov Inst. Math., 1999.
2. ______, *Sous-groupes paraboliques et représentations de groupes branchés*, submitted, 2000.
3. Gilbert Baumslag, *Topics in combinatorial group theory*, Lectures in Mathematics, ETH Zürich, Birkhäuser Verlag, Basel, 1993.
4. Jacek Fabrykowski and Narain D. Gupta, *On groups with sub-exponential growth functions*, J. Indian Math. Soc. (N.S.) **49** (1985), no. 3-4, 249–256.
5. Rostislav I. Grigorchuk, *On Burnside's problem on periodic groups*, Funktsional. Anal. i Prilozhen. **14** (1980), no. 1, 53–54, English translation: Functional Anal. Appl. **14** (1980), 41–43.
6. ______, *Just infinite branched groups*, Horizons in Profinite Groups (Dan Segal, Markus P. F. du Sautoy, and Aner Shalev, eds.), Birkhaüser, Basel, 2000, pp. 121–179.
7. Rostislav I. Grigorchuk and Andrzej Żuk, *On the asymptotic spectrum of random walks on infinite families of graphs*, Proceedings of the Conference "Random Walks and Discrete Potential Theory" (Cortona) (M. Picardello and W. Woess, eds.), Symposia Mathematica (Cambridge University Press), 22–28 June 1997, (to appear), pp. 134–150.
8. Narain D. Gupta and Said N. Sidki, *On the Burnside problem for periodic groups*, Math. Z. **182** (1983), 385–388.
9. ______, *Some infinite p-groups*, Algebra i Logika **22** (1983), no. 5, 584–589.
10. Pierre de la Harpe, A. Guyan Robertson, and Alain Valette, *On the spectrum of the sum of generators for a finitely generated group*, Israel J. Math. **81** (1993), no. 1-2, 65–96.
11. Harry Kesten, *Symmetric random walks on groups*, Trans. Amer. Math. Soc. **92** (1959), 336–354.
12. Alexander Lubotzky, *Cayley graphs: Eigenvalues, expanders and random walks*, Surveys in combinatorics, 1995 (Stirling), Cambridge Univ. Press, Cambridge, 1995, pp. 155–189.
13. Jean-Pierre Serre, *Répartition asymptotique des valeurs propres de l'opérateur de Hecke T_p*, J. Amer. Math. Soc. **10** (1997), no. 1, 75–102.

14. John von Neumann, *Zur allgemeinen Theorie des Masses*, Fund. Math. **13** (1929), 73–116 and 333, = *Collected works*, vol. I, pages 599–643.